MMP Learning Seminar.

Week 3~~3~~:

On termination of flips.

Shokurov and Birkar's approaches.

On termination of flips:

$$\left.\begin{array}{c}\text{Termination in dim } n-1 \\ + \\ \text{Existence of flips in dim } n-1\end{array}\right\} \Longrightarrow \text{Existence of flips in dim } n.$$

(Hacon–McKernan).

Log discrepancy:

(X,Δ) log pair. E prime divisor over X.

$\varphi: Y \longrightarrow X$ proj birational morphism extracting E,

then we can write $K_Y + \Delta_Y = \varphi^*(K_X+\Delta)$.

We define the log discrepancy of (X,Δ) at E

$$a_E(X,\Delta) = 1 - \operatorname{coeff}_E(\Delta_Y).$$

(X,Δ) is klt if $a_E(X,\Delta) > 0$ for all E

(X,Δ) is lc if $a_E(X,\Delta) \geq 0$ for all E.

Minimal log discrepancy:

$$\mathrm{mld}(X,\Delta; x) = \min\{a_E(X,\Delta) \mid c_X(E) = x\}$$

Examples: $\mathrm{mld}(\mathbb{A}^n; 0) = n$

X_n has a A_n-sing at x, $x^2+y^2+z^{n-1}=0$.

Then $\mathrm{mld}(X_n; x) = 1$.

$\mathrm{mld}(C_n; v) = \frac{2}{n}$, where C_n is the cone over a rat curve of degree n

$\mathrm{mld}(C_e; v) = 0$, where C_e is the cone over an elliptic curve

we will use this invariant to study flips.

Lemma (Monotonicity): Let

$$\begin{tikzcd}
(X,\Delta) \arrow[rr, dashed, "\pi"] \arrow[dr, "\varphi"'] & & (X^+,\Delta^+) \arrow[dl, "\varphi^+"] \\
& W &
\end{tikzcd}$$

be a flip for $K_X+\Delta$. For every E over X, we have

$$a_E(X,\Delta) \leq a_E(X^+,\Delta^+)$$

Furthermore, the inequality is strict iff $c_X(E) \subseteq \mathrm{Ex}(\pi)$.

Definition: $f: X \dashrightarrow X^+$ is a d-contraction if the complete transform of each irreducible subvariety of $\dim \geq n-d$ is well-defined and its image have the same dim or $\dim \leq n-d-1$.

n-contraction = regular morphism.

1-contraction = birational contraction.

bi-d-birational : f and f^{-1} are d-contractions

Lemma: Any sequence of d-contractions $X_i \dashrightarrow X_{i+1}$ of projective varieties birationally d-stabilizes, that is, for $i \gg 0$, they are all bi-d-birational.

Idea: Instead of looking at $\rho(X)$ $(d=1)$, we consider the rank of the group of alg cycles of codim d modulo alg equiv.

Conjectures on minimal log discrepancies:

Conjecture (Shokurov, 2000; ACC): Let n be a positive integer. Let $\Lambda \subseteq \mathbb{R}$ be a set satisfying the descending chain condition (DCC). Then the set

$$\{ \mathrm{mld}\,(X, \Delta; x) \mid (X,\Delta) \text{ } n\text{-dim klt}, \ \mathrm{coeff}\,(\Delta) \subseteq \Lambda \}.$$

satisfies the ascending chain condition.

There is an upper bound for the mld of n-dim.

Conjecture (Ambro, 2000s, LSC): (X,Δ) lc pair. $x \in X$ be a d-dimensional point. There exists $U \subseteq X$ so that for every d-dim point x' for which $\overline{x'} \cap U \neq \emptyset$, we have:

$$\mathrm{mld}(X,\Delta; x) \leq \mathrm{mld}\,(X,\Delta; x').$$

The upper bound is n.

Lemma: (X,Δ) log pair. There exists a finite partition X_i of X, so that each X_i is constructible and mld function is constant on $(X_i)_d$ for each i and d.

mld stratification is constructible

Theorem (Shokurov, 2004): ACC in dim n + LSC in dim n

$\Longrightarrow$ Termination in dim n

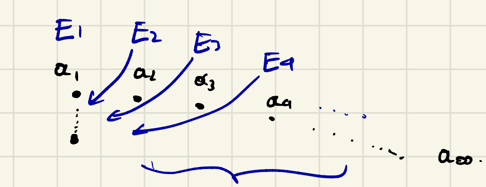
Sketch of the proof:

$$
\begin{tikzcd}
(X_1,\Delta_1) \arrow[rr, dashed, "\pi_1"] \arrow[dr, "\varphi_1"'] & & (X_2,\Delta_2) \arrow[dl, "\varphi_1^+"] \arrow[rr, dashed, "\pi_2"] \arrow[dr, "\varphi_2"'] & & (X_3,\Delta_3) \arrow[dl, "\varphi_2^+"] \arrow[r, dashed, "\pi_3"] & {} \\
& W_1 & & W_2 & &
\end{tikzcd}
$$

$$a_i = \mathrm{mld}\,(X_i, \Delta_i\,;\, Ex(\pi_i)) \geq 0.$$

Step 1: The minimum a_i stabilizes.

There exists $a \geq 0$, so that $a_i \geq a$ for every i and $a = a_i$ for infinitely many i's.



(X_1,Δ_1)

$\mathbb{Z}_i[\frac{1}{p}]$ where p only depends on (X_1,Δ_1)

$\underline{a_{E_i}(X_1,\Delta_1)} \leq a_{E_i}(X_i,\Delta_i)$ by monotonicity.

Infinite increasing sequence? This would violate **ACC**.

α stabilizes and $a_i = \alpha$ for infinitely many i's

Step 2: The maximal dimensional center, wherein the mld $a_i = \alpha$ is attained stabilizes. We call it d.

Step 3. On each (X_i, Δ_i) there exists a closed subvariety $W_i \subseteq X_i$ for which the following holds:

1) Each d-point x with $\mathrm{mld}(X_i, \Delta_i; x) = \alpha$ belongs to W_i,
2) each d-point $x \in W_i$ has $\mathrm{mld}(X_i, \Delta_i; x) \leq \alpha$, and
3) each generic d-point $x \in W_i$ has $\mathrm{mld}(X_i, \Delta_i; x) = \alpha$

This follows from stratification Lemma + **LSC**.

Step 4: W_{i+1} is the proper transform of W_i.

Step 5: $W_i \dashrightarrow W_{i+1}$ stabilizes birational

Step 6: The transformation $W_i \dashrightarrow W_{i+1}$ are

$m = \dim W_i$.

Step 6: The transformation $W_i \dashrightarrow W_{i+1}$ are birational $(m-d)$-contractions. Furthermore, at least one d-point is contracted whenever $a_i = a$, and there exists a point x in $\mathrm{Ex}(\pi_i)$ with $\dim x = d$ and $\mathrm{mld}(X_i, \Delta_i ; x) = a$.

infinitely many flips with this condition

This follows from Step 3 + Monotonicity.

eventually is bi-$(m-d)$-birational $\square$.

Philosophy of the previous proof.

$$(X_1, \Delta_1) \overset{\pi_1}{\dashrightarrow} (X_2, \Delta_2) \overset{\pi_2}{\dashrightarrow} (X_3, \Delta_3) \dashrightarrow \dots$$

W_i = locus where the mld of (X_i, Δ_i) is computed.

If $W_i \subseteq \mathrm{Ex}(\pi_j)$, then $\underline{\mathrm{mld}(X_j, \Delta_j) > \mathrm{mld}(X_i, \Delta_i)}$, ✓

- We used LSC to prove that, $\underline{W_i \text{ is eventually flipped}}$, or all $\underline{\text{flips are eventually disjoint from } W_i}$. → mld increases
- We use ACC and the previous dot to get a contradiction

Log canonical thresholds:

(X, Δ) klt, $E \geq 0$ on X $\mathbb{Q}$-Cartier.

We define $\operatorname{lct}((X,\Delta); E) = \sup\{t \mid (X, \Delta + tE) \text{ is lc}\}$

Examples: $\operatorname{lct}(\mathbb{A}^n, qH) = \frac{1}{q}$.

$$\operatorname{lct}(\mathbb{A}^2, \{x^3 - y^2 = 0\}) = \frac{5}{6}.$$

$H = V(x_1^{a_1} + \dots + x_n^{a_n}) \subseteq \mathbb{A}^n$, $\operatorname{lct}(\mathbb{A}^n, H) = \min\{1, \sum_{i=1}^n \frac{1}{a_i}\}$.

$1 - \frac{1}{n}$

Conjecture (ACC for lct's): Let Λ be a set satisfying the DCC. Let n be a positive integer. Then the set

$$LCT(\Lambda, n) := \left\{ \operatorname{lct}((X,\Delta); H) \;\middle|\; \begin{array}{c} (X,\Delta) \text{ lc } n\text{-dim}, \\ H \subseteq X \ \mathbb{Q}\text{-Cartier} \\ \operatorname{Coeff}(\Delta),\ \operatorname{coeff}(H) \subseteq \Lambda \end{array} \right\}$$

satisfies the ascending chain condition

Birkar's approach to termination:

(X,Δ) is an effective pair $K_X+\Delta \sim_{\mathbb{Q}} H \geq 0.$

$$(X_1,\Delta_1) \overset{\pi_1}{\dashrightarrow} (X_2,\Delta_2) \overset{\pi_2}{\dashrightarrow} (X_3,\Delta_3) \overset{\pi_3}{\dashrightarrow} \cdots$$

$H_i = \pi_{i*} H_{i-1}$ inductively.

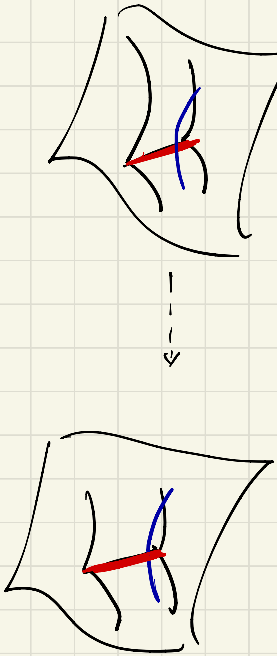
For each i, we can define $\lambda_i = \mathrm{lct}((X_i,\Delta_i); H_i) \geq 0.$

$K_{X_i}+\Delta_i \sim_{\mathbb{Q}} H_i \geq 0.$ Then.

$$K_{X_i}+\Delta_i+\lambda H_i \sim_{\mathbb{Q}} (1+\lambda)(K_{X_i}+\Delta_i)$$

A flip for $K_{X_i}+\Delta_i$ is also a flip for $K_{X_i}+\Delta_i+\lambda H_i.$

Question: When does $\lambda_i > \lambda_{i-1}$?



(X_i, Δ_i)

H_i

= flipping locus.

$\lambda_i = \mathrm{lct}((X_i, \Delta_i); H_i)$.

$= \mathrm{lcc}(X_i, \Delta_i + \lambda_i H_i)$

(X_{i+1}, Δ_{i+1})

H_{i+1}

$\lambda_i = \lambda_{i+1}$

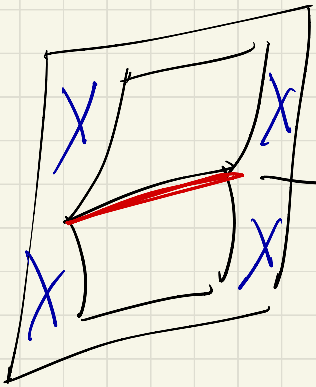
Remark: If the flipping loci does not contain the lcc

then $\lambda_{i+1} = \lambda_i$.

If the flipping loci does contain the lcc

then $\lambda_{i+1} > \lambda_i$

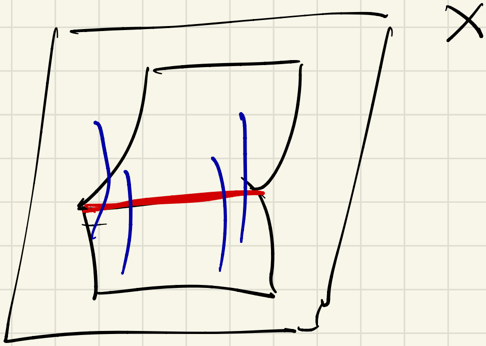
only finitely many times.

$X =$  $\mathrm{lcc} = Z$

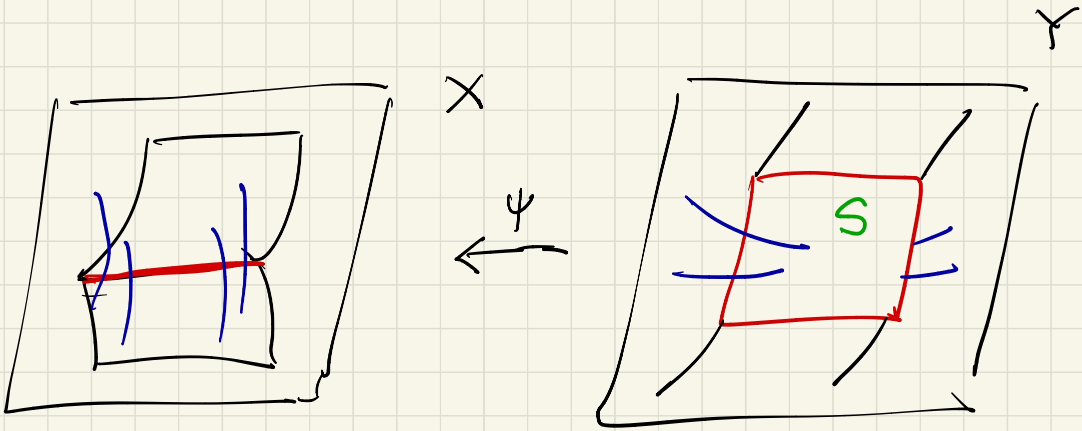
Replace X with $X \setminus Z$.

$\mathrm{lct}((X, \Delta); H) < \mathrm{lct}((X \setminus Z, \Delta|_{X \setminus Z}); H|_{X \setminus Z})$.

can happen only finitely many times.



$$Z = \mathrm{lcc}((X,\Delta); H).$$



In the model Y, the red loci appears with coeff one in the boundary of

$$\psi^*(K_X+\Delta+\lambda H) = K_Y + S + \dots$$

i) We lift the sequence of flips from X to Y.

ii) We perform adj to S and obtain a $\rightarrow\leftarrow$ by lower dim term

Theorem (Birkar 2007):

Assume ACC for lct's in dim n

$$+$$

$$\underbrace{\text{Assume termination of } \boxed{\text{lower dim flips } (\leq n-1)}}$$

$$\Downarrow$$

Termination of flips for n-dim $\boxed{\text{effective pairs.}}$

mld.

lct

Summary:

LSC

Try to study the most sing loci. of sequence of flips. Adjunction

the sequence must terminate around the most sing loci and

the most sing loci can change only finitely many times

ACC for mld's

ACC for lct's.

In BCHM:

$K_X+\Delta$ is big, we have a lot of sections

$H_1, \dots, H_n \in |K_X+\Delta|_{\mathbb{Q}}$, we can produce

a lot of thresholds:

$$\operatorname{lct}((X,\Delta); \lambda_1 H_1 + \dots + \lambda_n H_n).$$

For certain choices of this thresholds, Z

will again be of general type.

Then, flips should terminate over Z. $\square$